MBW-003-1164007 Seat No. ________

M. Sc. (Mathematics) (Sem. IV) (CBCS) Examination

April / May - 2018

EMT - 4031 : Commutative Ring Theory

(New Course)

Faculty Code : 003
Subject Code : 1164007

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) There are five questions.
(2) All questions are compulsory.
(3) Each question carries 14 marks.

1 Fill in the blanks : (Each question carries two marks) **14**

(1) When is a property of a ring said to be a local property?

(2) Define Artinian ring.

(3) Define a Noetherian Module.

(4) Define a ring homomorphism. Verify that any ring homomorphism from a field F into a nonzero ring is bijective.

(5) Define Jacobson radical of a ring R.

(6) Define a primary ideal of a ring R.

(7) Prove that 1 and –1 are the only units in $\mathbb{Z}$.

2 Attempt any **two** : **14**

(a) Prove that the nilradical of a ring R is the intersection of all the prime ideals of R. **7**

(b) Let M be a module over a ring R. Prove that M is finitely generated if and only if M is isomorphic to a quotient of R^n for some $n \in \mathbb{N}$. **7**

(c) (i) Let $P_1, \ldots\ldots, P_n$ be prime ideals of a ring R. If an ideal of R is such that $I \subseteq \bigcup_{i=1}^{n} P_i$, then show that $I \subseteq P_i$, for some $i \in \{1, 2, \ldots\ldots, n\}$. **7**

(ii) Let $I_1, I_2, \ldots\ldots, I_n$ be ideals of a ring R. If a prime ideal P of R satisfies $P \supseteq \bigcap_{i=1}^{n} I_i$, then prove that $P \supseteq I_i$, for some $i \in \{1, 2, \ldots\ldots, n\}$.

3 All compulsory : **14**

(a) Let P be any nonzero prime ideal of a principal ideal domain R. Prove that P is a maximal ideal of R. **5**

(b) Let M_1, M_2 be submodules of a moducle M over a ring R. Show that $\frac{M_1 + M_2}{M_1} \cong \frac{M_2}{M_1 \cap M_2}$ as R–modules. **5**

(c) Let I be an ideal of a ring R. Prove that $r(r(I)) = r(I)$. **4**

OR

3 All compulsory : **14**

(a) Let S be a multiplicatively closed subset of a ring R. For any ideal I of R, prove that $S^{-1}(\sqrt{I}) = \sqrt{S^{-1}I}$. **4**

(b) Let R be a subring of a ring T. If $t \in T$ is integral over R, then prove that $R[t]$ is finitely generated R–module. **5**

(c) Let S be a multiplicatively closed subset of a ring R. Let $g : R \to T$ be a ring homomorphism such that $g(s)$ is a unit in T for all $s \in S$. Prove that there exists a unique ring homomorphism $h : S^{-1}R \to T$ such that $g(r) = h\left(\frac{r}{1}\right)$, for all $r \in R$. **5**

4 Attempt any **two** : **14**

(a) State and prove first uniqueness theorem on decomposable ideals in a ring R. **7**

(b) Let M be a module over a ring R. Show that M satisfies ascending chain condition on submodules if and only if every submodule of M is finitely generated. **7**

(c) Let R be a Noetherian ring. If I is a proper irreducible ideal of R, then prove that I is primary. **7**

5 Attempt any **two** : **14**

(a) State and prove Chinese remainder theorem. **7**

(b) Prove that the nilradical of an Artin ring R is nilpotent. **7**

(c) State and prove Nakayama's lemma. **7**